



## Modifying the convexity condition in Data Envelopment Analysis (DEA)

### Modificación de la condición de convexidad en el Análisis Envoltente de Datos (AED)

Roghyeh Malekii Vishkaei<sup>1</sup>, Behrouz Daneshian<sup>2\*</sup>, Farhad Hosseinzadeh Lotfi<sup>3</sup>

<sup>1</sup> Department of Mathematics, Lahijan Branch, Islamic Azad University, Lahijan, Iran.

<sup>2</sup> Department of Mathematics, Central Tehran Branch, Islamic Azad University, Tehran, Iran.

<sup>3</sup> Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran.

Corresponding author Email: [be\\_daneshian@yahoo.com](mailto:be_daneshian@yahoo.com)

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#### ABSTRACT

Conventional Data Envelopment Analysis (DEA) models are based on a production possibility set (PPS) that satisfies various postulates. Extension or modification of these axioms leads to different DEA models. In this paper, our focus concentrates on the convexity axiom, leaving the other axioms unmodified. Modifying or extending the convexity condition can lead to a different PPS. This adaptation is followed by a two-step procedure to evaluate the efficiency of a unit based on the resulting PPS. The proposed frontier is located between two standard, well-known DEA frontiers. The model presented can differentiate between units more finely than the standard variable return to scale (VRS) model. In order to illustrate the strengths of the proposed model, a real data set describing Iranian banks was employed. The results show that this alternative model outperforms the standard VRS model and increases the discrimination power of (VRS) models.

**Keywords:** Data Envelopment Analysis (DEA), Convexity condition, Constant return to scale (CRS), Variable return to scale (VRS), Efficiency estimation.

#### RESUMEN

Los modelos de análisis envoltente de datos convencionales (DEA) se basan en un conjunto de posibilidades de producción (PPS) que satisface varios postulados. La extensión o modificación de estos axiomas conduce a diferentes modelos DEA. En este artículo, nuestro enfoque se concentra en el axioma de convexidad, dejando los otros axiomas sin modificar. Modificar o extender la condición de convexidad puede conducir a un PPS diferente. A esta adaptación le sigue un procedimiento de dos pasos para evaluar la eficiencia de una unidad en función del PPS resultante. La frontera propuesta está ubicada entre dos fronteras de la DEA estándar y conocidas. El modelo presentado puede diferenciar entre unidades con mayor precisión que el modelo de retorno a escala variable estándar (VRS). Para ilustrar las fortalezas del

modelo propuesto, se utilizó un conjunto de datos reales que describen los bancos iraníes. Los resultados muestran que este modelo alternativo supera al modelo estándar de VRS y aumenta el poder de discriminación de los modelos (VRS).

**Palabras clave:** Análisis envolvente de datos (ADE), Condición de convexidad, Retorno constante a escala (CRS), Retorno variable a escala (VRS), Estimación de eficiencia.

## 1. INTRODUCTION

In standard microeconomic theory, the concept of a production function forms the basis for a description of input-output relationships in a firm. The production function shows the maximum amount of outputs that can be achieved by combining various quantities of inputs. Thus initially the problem is the construction of an empirical production function or frontier based on the observed data. Data envelopment analysis (DEA) has been recognized as an effective nonparametric mathematical optimization technique for measuring the relative efficiency of a group of comparable decision-making units (DMUs) with multiple inputs and outputs (Charnes et al., 1994), DEA has been used in many contexts including education systems, health care units, agricultural productions, military logistics and many other applications (Charnes et al., 1994; Alder et al., 2006; Emrouznejad et al., 2008; Lozano et al., 2013; Asmild et al., 2013). Conventional DEA models assume real-valued and non-negative inputs and outputs vectors. Besides, each DMU is expressed by the notation  $(x, y)$ . The first component is regarded to input and the second can specify outputs. The set of feasible activities is called the production possibility set (PPS) and is denoted by  $P$  and satisfied in the axioms of Envelopment, free disposability, constant return to scale (CRS) or unbounded ray and semi positive linear combination of activities to construct  $P$ . Regarding to these axioms, the ray from the origin through the highest point is the CRS efficient frontier. As pointed out correctly in Podinovski, 2004 CRS models require full proportionality assumptions between all inputs and outputs. The pioneering CCR<sup>1</sup> model proposed by Charnes et al., 1978 satisfied in the above mentioned postulates. Various extension of axioms has been proposed in DEA literature. Among the modified variation, BCC<sup>2</sup> model was presented by Banker, 1984. The BCC model has its production frontiers spanned by the convex hull of the existing DMUs. In other words, the axiom of CRS (Constant Return to Scale) was extended and modified. The frontier has piecewise linear and concave characteristics. Also satisfies in variable return to scale (VRS) properties. Other extension and modification of CRS property can be found in (Seiford et al., 1990; Petersen., 1990; Bogetoft, 1996; Bogetoft et al., 2000; Färe et al., 1985) Theoretically, these classic models evaluate proportional efficiency by maximizing ratio of the weighted sum of its outputs to the weighted sum of its inputs, subject to the condition that this ratio does not exceed one for any DMU. Another variation of DEA models is Additive model. This model has the same production possibility set as CCR and BCC models and their variants but treats the input excesses and output shortfalls directly in the objective function. A slack-based measure of efficiency (SBM) is another version of DEA models. This measure makes its efficiency evaluation as effected in objective, invariant to the measures used for different inputs and outputs (Cooper et al., 2007). Free disposal Hull (FDH) model assumes a nonconvex possibility set and firstly presented by Deprins et al., 2006. This model ignores the convexity axioms and then has extended by Tulkens, 2006 As it can be seen the modification and extensions of axioms leads to different models. As another example, modifying free disposability leads to notation of weak disposability which was demonstrated by Kuosmanen, 2005 and extended by various authors. Among them, refer to (Podinovski et al., 2011; Färe et al., 2004) . For the notation of congestion can refer to Cherchye et al., 2001 As it can be seen different setting of axioms lead to different models. One of the most modified axioms was convexity. When the attention restricts to convexity axioms, the most applied BCC model is imagined. In this model, the convex hull of existing

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<sup>1</sup> CCR (Charnes, Cooper and Rhodes)

<sup>2</sup> BCC (Banker, Charnes and Cooper)

units' parallel to variable return to scale (VRS) characteristic evaluates the efficiency. Reserving the other axioms, that is to say, Envelopment, free disposability and VRS, a dilemma can raise. If the construction of convex hull is modified, how the efficiency frontier changes. What's more, to what extent, the efficiency score does change? This article tackles with this dilemma. Looking over the existing studies, this paper focuses on DEA axioms while the convexity axioms is modified or relaxed. In other words, the postulates of Envelopment, free disposability also variable return to scale are preserved. On the other hand, the convexity condition has been modified and extended. This extension is imposed as an interval with upper and lower bound. Equipped with the relaxed axioms, a two-step procedure is presented to evaluate the efficiency. The first stage surveys the minimum quantity of boundaries which might be used in the second stage. A linear model is proposed to evaluate the efficiency score in the second stage. As the relaxed axioms claim the frontier of proposed method can be inserted between two classic DEA model, i.e. CCR and BCC model. The results show that this model can estimate the efficiency of DMUs more accurate than BCC model and operates as well as this model. The real example of 37 Iranian bank branches supports the idea behind this modified axiom. The rest of this paper is organized as follows: In the following section, the axioms of standard DEA models are presented. A proposed two-phase model with modified convexity condition is presented in the third section. An empirical example highlights the model strengths in section 4. Conclusion will end the paper.

## 2. PRELIMINARIES

In DEA each observed DMU is characterized by a pair of non-negative input and output vectors  $(X_j, Y_j) \in R^{m+s}$ ,  $j \in J = \{1, \dots, n\}$ . The classic (Charnes et al., 1978) DEA model assumes that the underlying production possibility set (PPS) denoted by  $T = \{(x, y) \mid x \in R^m \text{ can produce } y \in R^s\}$  and satisfies the following axioms:

- 1) Envelopment:  $(X_j, Y_j) \in T, \forall j \in J$ .
- 2) Free disposability:  $(x, y) \in T, (u, v) \in R^{m+s}, y \geq v$  then  $(x+u, y-v) \in T$ .
- 3) Constant Return to Scale:  $(x, y) \in T \Rightarrow (\lambda x, \lambda y) \in T, \forall \lambda \in R^+$ .
- 4) Convexity:  $(x, y), (\bar{x}, \bar{y}) \in T \Rightarrow (\hat{x}, \hat{y}) = \lambda(x, y) + (1-\lambda)(\bar{x}, \bar{y}) \in T \quad \forall 0 \leq \lambda \leq 1$ .

According to the minimum extrapolation principle (Banker et al., 1984), the DEA production possibility set (PPS) is the intersection of all sets  $S \subset R^{m+s}$  that satisfy the maintained axioms. Under the maintained assumptions (1) – (4), the minimum extrapolation PPS can be explicitly stated as:

$$T_{CRS}^{DEA} = \{(x, y) \mid x \geq \sum_{j=1}^n \lambda_j X_j, y \leq \sum_{j=1}^n \lambda_j Y_j, \lambda \geq 0\}.$$

If the study turns to efficiency estimations of DMUs and attention has restricted to classic Farrell input efficiency measurement defined as:

$$Eff(x_o, y_o) = \min\{\theta \mid (\theta x_o, y_o) \in T\}$$

Where vector  $(x_o, y_o)$  refers to the observed or hypothetical DMU under evaluation. Applying this measure directly to  $T_{CRS}$  can yield a monotonic and convex set of points. Also, the mathematical

programming problem of this evaluation is called CCR (Charnes et al., 1978) input-oriented and has the following format:

$$\begin{aligned}
 \text{Min} \quad & \theta_o \\
 s.t.: \quad & \sum_{j=1}^n \lambda_j x_{ij} \leq \theta x_{io} \quad i=1, \dots, m \\
 & \sum_{j=1}^n \lambda_j y_{rj} \geq y_{ro} \quad r=1, \dots, s \\
 & \lambda_j \geq 0 \quad j=1, \dots, n
 \end{aligned} \tag{1}$$

This model is a constant return to scale (CRS) program and it assumes that the status of all input/output variables are known prior to solving the model. The efficiency ratio  $\theta_o$  ranges between zero and one, with  $DMU_o$  being considered relatively efficient if it receives a score of one. From a managerial perspective, this model delivers assessments and targets with an output maximization orientation. In addition to these basic postulates, technology  $T$  also be assumed to satisfy some of the following returns to scale axioms:

5. Non-Increasing return to scale (NIRS):  $(x, y) \in T$  and  $\lambda \in [0, 1]$  then  $(\lambda x, \lambda y) \in T$ .
6. Non-Decreasing return to scale (NDRS):  $(x, y) \in T$  and  $\lambda \geq 1$  then  $(\lambda x, \lambda y) \in T$ .

Under the minimal assumptions of free disposability (2) and convexity (4), the minimum extrapolation PPS is referred to as the variable return to scale (VRS) technology, formally stated as:

$$T_{VRS}^{DEA} = \left\{ (x, y) \left| x \geq \sum_{j=1}^n \lambda_j X_j, y \leq \sum_{j=1}^n \lambda_j Y_j, \sum_{j=1}^n \lambda_j = 1, \lambda_j \geq 0 \right. \right\}$$

Regarding to this technology, the mathematical programming problem for estimating the efficiency of DMUs has the following format:

$$\begin{aligned}
 \text{Min} \quad & \theta \\
 s.t.: \quad & \sum_{j=1}^n \lambda_j x_{ij} \leq \theta x_{io} \quad i=1, \dots, m \\
 & \sum_{j=1}^n \lambda_j y_{rj} \geq y_{ro} \quad r=1, \dots, s \\
 & \sum_{j=1}^n \lambda_j = 1 \\
 & \lambda_j \geq 0 \quad j=1, \dots, n
 \end{aligned} \tag{2}$$

The above model is an input oriented BCC (Banker et al., 1984) model. A similar model can be formulated to present output orientation VRS technology. If in addition to axioms (2) and (4) the assumption of NIRS (5) is imposed, the resulting minimum extrapolation PPS is the NIRS technology:

$$T_{NIRS}^{DEA} = \{(x, y) \mid x \geq \sum_{j=1}^n \lambda_j X_j, y \leq \sum_{j=1}^n \lambda_j Y_j, \sum_{j=1}^n \lambda_j \leq 1, \lambda \geq 0\}$$

On the other hand, if instead of (5) one assumes NDRS (6), then the minimum extrapolation PPS is the NDRS technology:

$$T_{NDRS}^{DEA} = \{(x, y) \mid x \geq \sum_{j=1}^n \lambda_j X_j, y \leq \sum_{j=1}^n \lambda_j Y_j, \sum_{j=1}^n \lambda_j \geq 1, \lambda \geq 0\}$$

As it can be seen relaxation of axiom (3) leads to models of variable and non-increasing decreasing) returns to scale (Seiford et al., 1990) In addition to these four classical DEA technology, many variations of axioms (2) -(4) have been presented in the literature. Relaxation of axiom (2) leads to models of weak disposability (Kuosmanen, 2005) and congestion (Cherchye et al., 2001). Relaxation of (4) leads to free disposable hull (Deprins et al., 2006) and free replicable hull models (Tulkens, 2006). The above technologies and models are common models of relaxation axioms. In following section relaxation of convexity axiom will be discussed from another perspective.

### 3. METHODOLOGY

Suppose that we have  $n$  DMUs and each  $DMU_j (j \in J = \{1, \dots, n\})$  uses  $m$  inputs  $x_{ij} (i = 1, \dots, m)$  to produce  $s$  outputs  $y_{rj} (r = 1, \dots, s)$ . Let  $T_{CRS}^{DEA}$  be the underlying production possibility set (PPS) satisfies the following axioms:

- 1) Envelopment of observed data:  $(x_j, y_j) \in T, \forall j \in J$ .
- 2) Free disposability:  $(x, y) \in T, x' \geq x$  and  $y' \leq y$  implies that  $(x', y') \in T$
- 3) Constant return to scale:  $(x, y) \in T \Rightarrow (\lambda x, \lambda y) \in T, \forall \lambda \in R^+$
- 4) Convexity:  $(x, y), (\bar{x}, \bar{y}) \in T \Rightarrow (\hat{x}, \hat{y}) = \lambda(x, y) + (1 - \lambda)(\bar{x}, \bar{y}) \in T \quad \forall 0 \leq \lambda \leq 1$
- 5) Minimal extrapolation: For each  $T$  satisfying in axioms 1-4, we have  $T \subset T'$

An Algebraic representation of the PPS for technology  $T_{CRS}^{DEA}$ , which satisfying the axioms 1–5, is given as

$$T_{CRS}^{DEA} = \{(x, y) \mid x \geq \sum_{j=1}^n \lambda_j X_j, y \leq \sum_{j=1}^n \lambda_j Y_j, \lambda \geq 0\}$$

Relaxation the convexity condition, the variable return to scale (VRS) technology was presented by Banker et al., 1984 and formally stated as:

$$T_{VRS}^{DEA} = \{(x, y) \mid x \geq \sum_{j=1}^n \lambda_j X_j, y \leq \sum_{j=1}^n \lambda_j Y_j, \sum_{j=1}^n \lambda_j = 1, \lambda \geq 0\}$$

The variable  $\lambda$  presents the non-negative intensity variable. It is worth to relax the convexity condition as a mutual equation. In other words, instead of the employing the relation  $\sum_{j=1}^n \lambda_j = 1$ , the summation of intensity variable can be inserted in an interval with variable boundaries. Without the loss of generality, the lower and upper bound can be defined as  $[1-\alpha, 1+\alpha]$ . That is to say  $1-\alpha \leq \sum_{j=1}^n \lambda_j \leq 1+\alpha$ . Interestingly,  $\alpha = 0$  is consistent with the variable return to scale technology  $T_{VRS}^{DEA}$ . In the spirit of relaxing convexity axiom, the goal comes to specification of  $\alpha$ . Toward this end, a two-phase procedure is established.

### First step:

Looking for the minimum quantity for variable  $\alpha$  falls behind the scope of proposed first stage.

Admittedly, the notation  $1-\alpha \leq \sum_{j=1}^n \lambda_j \leq 1+\alpha$  is a new variation of variable constant return to scale axiom. In the spirit of convexity relaxation, one could try to construct the PPS by employing the envelopment, free disposability and variable return to scale (VRS) axioms. The technology  $T_{RVRS}$  can be stated as follows:

$$T_{RVRS} = \{(x, y) \mid x \geq \sum_{j=1}^n \lambda_j X_j, y \leq \sum_{j=1}^n \lambda_j Y_j, 1-\alpha < \sum_{j=1}^n \lambda_j < 1+\alpha, \lambda \geq 0\}$$

It is worth to note that the proposed technology has the minimum extrapolation interpretation under adapted set of axioms. To measure efficiency improvement, a modified input efficiency measure is needed. Our attention has been restricted to Farrell input efficiency measure defined as:

$$Eff(x_o, y_o) = \min\{\theta + \varepsilon\alpha \mid (\theta x_o, y_o) \in T_{RVRS}\}$$

The unit under evaluation is denoted by  $(x_o, y_o)$ . Applying this measure directly to  $T_{RVRS}$  can yield the modified input efficiency scores relative to the adapted reference technology. Also it can be computed by solving the following model and solver software.

$$\text{Min} \quad \theta + \varepsilon\alpha$$

$$\text{s.t:} \quad \sum_{j=1}^n \lambda_j x_{ij} \leq \theta x_{io} \quad i = 1, \dots, m \quad (3-1)$$

$$\sum_{j=1}^n \lambda_j y_{rj} \geq y_{ro} \quad r = 1, \dots, s \quad (3-2) \quad (3)$$

$$1-\alpha \leq \sum_{j=1}^n \lambda_j \leq 1+\alpha \quad (3-3)$$

$$\lambda_j \geq 0 \quad j = 1, \dots, n$$

Symbol  $\varepsilon$  denotes a non-Archimedean infinitesimal and parameter  $\theta$  plays the role of abatement factor on inputs, as it does on model (2). One important feature of model (3) is the last constraint. This constraint

imposes convexity employing a mutual relation. Hence, the aim is to probe for a model that selects minimum statue for boundaries. For doing so, the objective function has set as  $\theta + \varepsilon\alpha$ . In essence, this modified measure gauges' efficiency in the radial fashion relative to the monotonic hull of the PPS. Model (3) selects a minimum statue for boundaries hence model feasibility and optimality has been guaranteed.

**Theorem1:** Model (3) is always feasible.

**Proof:** since  $(\alpha = 0, \lambda_j (j \neq o) = 0, \lambda_o = 1)$  is a feasible solution of model (3). So the model is always feasible. So, the first step of the proposed method is tended to solve model (3).

**Second step:**

Equipped with the optimal solutions of model (3) in the first phase, let  $\alpha' = \text{Min}\{\alpha_i : i = 1, \dots, n, \alpha_i \neq 0\}$ . Regarding the axioms of envelopment, free disposability, variable return to scale (VRS) and relaxed convexity, we propose the following model for measuring the efficiency of  $DMU_o$ .

$$\begin{aligned} \text{Min} \quad & \theta \\ \text{s.t.} \quad & \sum_{j=1}^n \lambda_j x_{ij} \leq \theta x_{io} \quad i = 1, \dots, m \end{aligned} \quad (4-1)$$

$$\sum_{j=1}^n \lambda_j y_{rj} \geq y_{ro} \quad r = 1, \dots, s \quad (4-2) \quad (4)$$

$$1 - \alpha' \leq \sum_{j=1}^n \lambda_j \leq 1 + \alpha' \quad (4-3)$$

$$\lambda_j \geq 0 \quad j = 1, \dots, n$$

Model (4) differs from model (3) only with respect to the relaxed convexity constraints expressed as

$1 - \alpha' \leq \sum_{j=1}^n \lambda_j \leq 1 + \alpha'$  in case of the variable return to scale (VRS) specification. Additionally, the rest

constraints have the same feature. Likewise, parameter  $\theta$  plays the role of abatement factor as does in model (3). One important conclusion is that intensity variable is same in both models. This unchanged feature makes the proposed frontier acts between two known frontiers.

**Theorem2:** The efficiency score of model (4) is not worse than that of BCC model (model (2)). That is  $(\theta_{\text{BCC}} \geq \theta_{\text{NEW}})$ .

**Proof:**

The last constraint of model (4) claims  $1 - \alpha' \leq \sum_{j=1}^n \lambda_j \leq 1 + \alpha'$ . This mutual relation includes  $\sum_{j=1}^n \lambda_j = 1$ .

Let  $(\lambda, \theta)$  be a feasible solution of BCC model (model (2)), it is also a feasible solution of model (4). Comparing both models' feasible space, it is concluded that the space of BCC model (model (2)) is the subset of space of model (4). Hence, the optimal value of BCC model (model (2)) is not better than the optimal value of model (4). That is  $(\theta_{\text{BCC}} \geq \theta_{\text{NEW}})$ .

**Theorem 3:** In model (4), at least there exit one efficient unit.

**Proof:**

Consider the dual format of BCC model (model (2)):

$$\begin{aligned}
\text{Max} \quad & z = \sum_{r=1}^s u_r y_{rp} - u_{\circ} \\
s.t : \quad & \sum_{r=1}^s u_r y_{rj} - \sum_{i=1}^m v_i x_{ij} - u_{\circ} \leq 0 \quad j = 1, \dots, n \\
& \sum_{i=1}^m v_i x_{ip} = 1 \\
& v_i \geq 0 \quad i = 1, \dots, m \\
& u_r \geq 0 \quad r = 1, \dots, s
\end{aligned} \tag{5}$$

Let  $DMU_p(x_p, y_p)$  be under evaluated unit. If  $(u_p^*, v_p^*, u_{\circ p}^*)$  be the optimal solution of model (5), we have:

$$z_p^* = \sum_{r=1}^s u_r^* y_{rp} - u_{\circ p}^* = 1, \quad u_{\circ p}^* = 0 \tag{6}$$

Also, in optimality the first constraint of model (5) claims that:

$$\sum_{r=1}^s u_r^* y_{rp} - \sum_{i=1}^m v_i^* x_{ip} = 0 \tag{7}$$

The relation (7) admits that there must be at least one constraint for which the optimal weight  $(u^*, v^*)$  leads to equality between the left and right hand side. Now, consider the dual format of proposed model (4). The dual form is stated as follows:

$$\begin{aligned}
\text{Max} \quad & z = \sum_{r=1}^s u_r y_{rp} - (1 + \alpha)k + (1 - \alpha)k' \\
s.t : \quad & \sum_{r=1}^s u_r y_{rj} - \sum_{i=1}^m v_i x_{ij} - k + k' \leq 0 \quad j = 1, \dots, n \\
& \sum_{i=1}^m v_i x_{ip} = 1 \\
& v_i \geq 0 \quad i = 1, \dots, m \\
& u_r \geq 0 \quad r = 1, \dots, s \\
& k, k' \geq 0
\end{aligned} \tag{8}$$

Assume that  $(u_p^*, v_p^*, u_{\circ p}^*)$  be the optimal solution in the model (8). Let  $u_{\circ} = k - k'$ . Also, the first constraint is satisfied for the optimal solution. That is:

$$\sum_{r=1}^s u_p^* y_{rj} - \sum_{i=1}^m v_p^* x_{ij} - k + k' = 0$$

Since  $u_{\circ p}^* = 0$ , we have:



$$k - k' = 0 \Rightarrow k = k'$$

Clearly,  $k$  and  $k'$  are dual variables which corresponds to the constraint  $(4-3) 1 - \alpha' \leq \sum_{j=1}^n \lambda_j \leq 1 + \alpha'$ . In essence the acceptable value for dual variable  $k$  and  $k'$  can capture zero. That is  $k = k' = 0$ . Employing the complementary slackness theorem, we have:

$$\sum_{j=1}^n \lambda_j = 1 + \alpha'$$

$$\sum_{j=1}^n \lambda_j = 1 - \alpha'$$

Which is impossible. Therefore, the first constraint and the objective function of the dual model (8) will be as follows:

$$\sum_{r=1}^s u_r^* y_{rp} - \sum_{i=1}^m v_i^* x_{ip} = 0$$

$$z = \sum_{r=1}^s u_r^* y_{rp}$$

Then we have:

$$\sum_{i=1}^m v_i^* x_{ip} = 1$$

Finally:

$$z^* = \sum_{r=1}^s u_r^* y_{rp} = 1$$

Then, at least  $\theta_p^* = 1$  and this completes the proof.

Briefly, a two-stage approach can simplify the procedure of finding the minimum quantity of proposed lower and upper bound. Our next objective is to characterize PPS that satisfies the minimum extrapolation principle subject to the properties 1-4. To illustrate, consider a simple numerical example in a single-input single-output case. Table1 shows the five DMUs.

Table1. Data Set of five DMUs

DMU	Input	Output
D1	2	1
D2	3	4
D3	6	6
D4	9	7
D5	5	3

This example aptly illustrates that a sequential application of the axioms can generate a monotonic and convex hull of PPS. Figure 1 illustrates the example graphically. The reference technology  $T_{CRS}^{DEA}$ ,  $T_{VRS}^{DEA}$  and  $T_{RVRS}^{DEA}$  are verified.

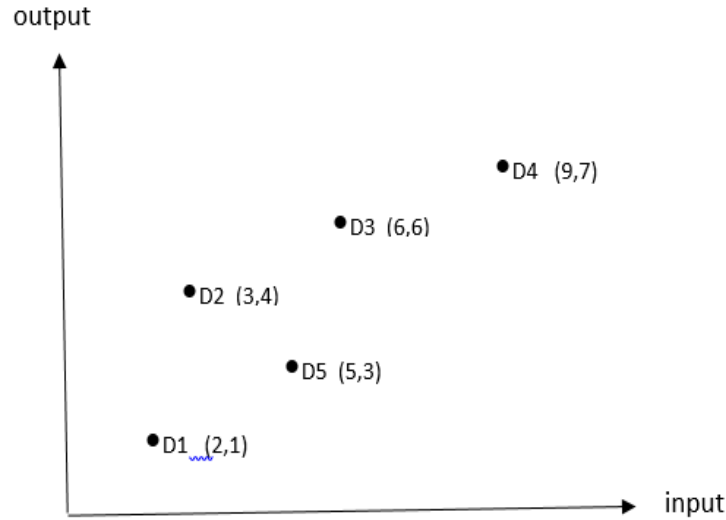


Figure 1. A simple case of 5 DMUs

The reference technology  $T_{CRS}^{DEA}$ ,  $T_{VRS}^{DEA}$  and  $T_{RVRS}^{DEA}$  are verified by this simple example. By applying envelopment, free disposability, constant return to scale and convexity parallel to minimum extrapolation the black linear frontier represents the DEA CRS frontier. The frontier was demonstrated as CCR frontier. Applying envelopment, free disposability and convexity again imposing variable return to scale parallel to minimum extrapolation. The piece-wise linear frontier depicts DEA VRS frontier. This frontier shows BCC frontier. For the purpose of comparison, the adapted axioms are implemented on this data set. Regarding to axiomatic foundation (envelopment, free disposability and variable return to scale) with relaxed convexity along with minimum extrapolation, the proposed frontier has the following format. Figure 2 represents the frontiers on the sample data set.

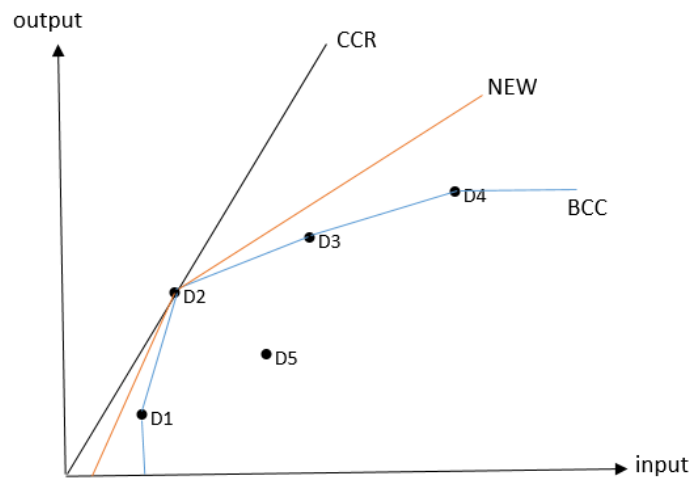


Figure 2. production possibility set (PPS) of CCR, BCC and proposed models

As Figure 2 shows the estimated frontier of model (4) have a character which comes closer to DEA CRS (CCR model) rather than DEA VRS (BCC model). What's more, it is worth to note that the proposed frontier has some similar behavior as DEAVRS (BCC model). This behavior can be characterized as the effect of constraint (4-3). Since this constraint allows the intensity variable  $\lambda$  alters in described optimal minimum bound. With reference to Figure 2, unit D2 has lied on three frontiers. Interestingly, the proposed frontier in unit D2 is tangent to the frontier between BCC and CCR frontier. This property is consistent with Theorem (3). One of the important feature of proposed model (model (4)) can be driven with regarding to Figure 2. The proposed frontier is upper or tangent to the DEA VRS frontier (BCC model). Therefore, the efficiency measured by model (4) is not worse than that measured by BCC model (model (2)). Clearly, this property supports Theorem2. The example illustrates the need to modify the input efficiency measure in the case of relaxation convexity. The radial efficiency scores obtained with our proposed model come close to those of DEA VRS (BCC) models, but there are some notable differences, particularly with DMUs 1, 3 and 4. DMU#1 is an efficient unit in DEA VRS model while its radial efficiency is 0.91 in proposed method. The benchmarks obtained by our proposed model are very close to the DEA VRS model. However, there exist notable differences between the computed targets in our proposed model with those obtained by DEA VRS formulation. Solving the DEA VRS model (model (2)), we obtain the radial efficiency of 0.53 and reference point (2.67, 3.01) with intensity variables  $\lambda_1 = 0.33$   $\lambda_2 = 0.67$  for inefficient unit D5. On the hand, as the Figure 2 shows the radial efficiency for this inefficient unit with regard to relaxed convexity technology is 0.46 and reference point (2.29, 2.27). This example demonstrates that proposed technology, which are justified by the adapted axiomatic analysis, can lead to efficiency scores and performance targets not greater that the known DEA VRS method. The application of the next section demonstrates that the proposed formulations can yield substantially different results.

#### 4. AN EMPIRICAL EXAMPLE

This section illustrates the proposed model in assessing 37 Iranian bank branches. Four factors are selected as inputs: personnel (Staff) privilege ( $x_1$ ), benefit payment ( $x_2$ ) and delayed demands ( $x_3$ ), and one factor recorded as output: interest ( $y_1$ ).

Table 2. data set of 37 Iranian bank branches

DMU	$x_1$	$x_2$	$x_3$	$y_1$
D01	46.79	18498995996	53264852560	19969314548
D02	24.51	14411686574	72380083269	15731542711
D03	15.51	8860736637	42598397319	5058977577
D04	24.26	13899053604	6411736105	12385243634
D05	30.65	28496201869	4366489880	21706793947
D06	25.54	37069157479	114935654544	14948666523
D07	48.23	19690080929	57234222760	44304276334
D08	33.73	22726721686	23456868289	13135578006
D09	54.48	38967409513	436938504803	84940713101
D10	40.34	56572978820	344245860744	77227782339
D11	18.43	48063580938	151405425096	175807481201
D12	31.11	13292920789	234489215832	44150867652
D13	40.58	115201941693	1310099546771	187612455426
D14	22.67	7507454431	65033329747	10959222029
D15	44.32	117212672954	128339685560	488528538701
D16	32.14	70423612922	575597667886	16490307548
D17	88.34	270906592344	818003444354	209621361454

D18	28.68	109088144925	794883602155	127427007609
D19	36.64	45435649108	161729833113	72598466849
D20	34.62	157467992263	445088568421	39162408384
D21	29.86	74974062462	59538234833	50822228758
D22	28.03	22146330179	535291913756	11752655320
D23	27.43	128232256516	214591474223	16784882367
D24	31.03	46101275529	100230783101	23404976771
D25	32.84	55605155762	167272021271	195548517193
D26	11.66	66337226010	47868669745	4659601911
D27	18.18	28831373265	19076494966	94858128176
D28	31.44	31145197431	346181300256	144133837620
D29	24.17	208373438089	20184723899	119202785808
D30	13.34	11370634434	44739871980	7332879040
D31	31.93	20645062576	46963006410	119719576835
D32	31.66	102960688131	66182377075	219739666495
D33	30.91	26446008607	231688742335	31935034820
D34	24.73	10076766017	71761767269	26146466300
D35	29.89	73594703492	85416663339	72319963500
D36	30.88	76524893150	16513150725	75638230244
D37	20.64	45687358659	4082761844	26851304512

For comparison, three alternative models were computed: CCR model (model (1)), BCC model (model (2)) and our proposed model (model (4)). The obtained radial input efficiency scores are presented in Table 3. Implementing our proposed model (4) might employ the optimal solutions of the first step model (3). Hence, we first run model (3) then model (4) is executed supposing

$\alpha'_j = \min\{\alpha_j \mid j=1,...,37, \alpha_j \neq 0\}$ . Therefore, by applying model (3) to the data in Table 2, we obtain the values of  $\alpha_j$   $j=1,...,37$ . Then in order to implement model (4) we obtain a value of  $\min\{\alpha_j \mid j=1,...,37, \alpha_j \neq 0\}$ , that is 0.12. This value is considered instead of  $\alpha'$  and implemented in model (4) and the model is run for data of Table 2.

Table 3. efficiency scores of the BCC, CCR and proposed models

	$\theta_{BCC}$	$\theta_{CCR}$	$\theta_{NEW}$
D01	0.62	0.19	0.56
D02	0.69	0.19	0.62
D03	1	0.1	0.89
D04	1	0.38	0.91
D05	1	0.86	0.96
D06	0.89	0.46	0.52
D07	0.68	0.39	0.62
D08	0.64	0.14	0.56
D09	0.47	0.39	0.44
D10	0.43	0.29	0.41
D11	1	0.99	0.99
D12	0.87	0.57	0.81
D13	0.48	0.42	0.47
D14	1	0.25	0.9
D15	1	1	1
D16	0.41	0.05	0.36
D17	0.26	0.24	0.24

D18	0.58	0.4	0.53
D19	0.5	0.34	0.46
D20	0.38	0.17	0.34
D21	0.52	0.21	0.46
D22	0.51	0.1	0.46
D23	0.45	0.08	0.4
D24	0.45	0.11	0.4
D25	0.83	0.8	0.82
D26	1	0.05	0.88
D27	1	1	1
D28	0.95	0.92	0.92
D29	1	1	1
D30	1	0.12	0.89
D31	1	1	1
D32	0.83	0.8	0.81
D33	0.54	0.23	0.49
D34	0.95	0.45	0.86
D35	0.54	0.23	0.48
D36	0.85	0.84	0.85
D37	1	1	1
AVE	0.73	0.47	0.68
VAR	0.06	0.12	0.06
STDEV	0.25	0.35	0.24

The last column of Table 3 shows the results of proposed method. The results of our proposed model come close to BCC model, but there are some notable differences, particularly units 30, 26, 11, 5, 4 and 3 are efficient in BCC model but in the proposed model are inefficient. Only six out of 37 units exactly give the same efficiency score as BCC model (2) does. In general, the efficiency scores of our proposed model (4) is always smaller than those of BCC model (2) and larger than those of CCR model (model (1)). This suggests that the relaxed convexity axioms can enhance the discriminatory power of the model. The application also demonstrates that the BCC model leads to overestimated efficiency assessments.

Finally, we would like to emphasize on statistical analysis. The last three rows of Table 3 depict average, variance and standard deviation respectively. The average of the proposed method (model (4)) is smaller than BBC model and larger than CCR model. That is to say, this average is inserted between two known models' average. This claims that the proposed model is able to identify the efficiency distribution. The variance of the proposed model (4) is same as the BCC model. Also, both of them are smaller than CCR variance quantity. What's more, the standard deviation of proposed model (4) is extremely small. This claims that the efficiency scores obtained by proposed model (4) tend to be close to the average. In practice, dispersion of efficiency scores in proposed model (4) is extremely lower than the CCR model. In other words, a low standard deviation indicates that the efficiency scores are spread out over a tighter range. Also, the results acknowledge that our proposed model (4) increase discrimination power of DEA VRS models.

## 5. CONCLUSION

Based on the widespread application of Data envelopment analysis (DEA) in performance estimation, it is worth to provide a suitable model to improve the efficiency. Each of the standard DEA models is constructed on specific postulates. Considering some axiomatic foundations, a production possibility set (PPS) is defined. In this paper we have presented an axiomatic foundation for a DEA model by relaxing the axiom of convexity. After modifying the notions of convexity, a two-step approach has been identified to yield a convex and monotone frontier. The proposed frontier has inserted between constant return to scale (CRS) DEA model and variable return to scale (VRS) DEA models. Although the efficiency score of

proposed model come on the average very close to those obtained by variable return to scale (VRS) DEA models, the differences can be rather substantial for benchmarking and target setting. An empirical efficiency evaluation of 37 bank branches further illustrated the importance of dealing with relaxed convexity axioms. Also, the application showed that the proposed model can increase the discrimination power of VRS models.

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